Out of equilibrium functional central limit theorems for a large network where customers join the shortest of several queues

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Abstract. Customers arrive at rate $N\alpha$ on a network of N single server infinite buffer queues, choose L queues uniformly, join the shortest one, and are served there in turn at rate β. We let N go to infinity. We prove a functional central limit theorem (CLT) for the tails of the empirical measures of the queue occupations, in a Hilbert space with the weak topology, with limit given by an Ornstein-Uhlenbeck process. The *a priori* assumption is that the initial data converge. This completes a recent functional CLT in equilibrium in Graham [3] for which convergence for the initial data was not known *a priori*, but was deduced *a posteriori* from the functional CLT.

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1 Introduction

1.1 The queuing model

We continue the asymptotic study for large N and fixed L initiated in Vvedenskaya et al. [9] of a Markovian network constituted of N single server infinite buffer queues. Customers arrive at rate $N\alpha$, are allocated L distinct queues uniformly at random, and join the shortest, ties being resolved uniformly at random. Service is at rate β. Arrivals, allocations and services are independent. The interaction structure depends on sampling from the empirical measure of L-tuples of queue states; in statistical mechanics terminology, this constitutes L-body mean-field interaction.

Let $X_i^N(t)$ be the length of queue i at time $t \ge 0$. The process $(X_i^N)_{1\le i\le N}$ is Markov, its empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}$ has samples in $\mathcal{P}(\mathbb{D}(\mathbb{R}_+,\mathbb{N}))$, and its marginal process $(\mu_t^N)_{t\ge 0}$ has sample paths in $\mathbb{D}(\mathbb{R}_+,\mathcal{P}(\mathbb{N}))$. We are interested in the tails of the marginals, and consider

$$\mathcal{V} = \{(v(k))_{k\in\mathbb{N}} : v(0) = 1,\ v(k) \ge v(k+1),\ \lim v = 0\}\ , \qquad \mathcal{V}^N = \mathcal{V} \cap \frac{1}{N}\mathbb{N}^{\mathbb{N}}\,,$$

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with the uniform topology (which coincides here with the product topology) and the process $R^N = (R_t^N)_{t\geq 0}$ with sample paths in $\mathbb{D}(\mathbb{R}_+, \mathcal{V}^N)$ given by

$$R_t^N(k) = \frac{1}{N}\sum_{i=1}^N \mathbb{I}_{X_i^N(t)\geq k}\,.$$

The processes $(\mu_t^N)_{t\geq 0}$ and $(R_t^N)_{t\geq 0}$ are in relation through $p \in \mathcal{P}(\mathbb{N}) \longleftrightarrow v \in \mathcal{V}$ for $v(k) = p[k,\infty)$ and $p\{k\} = v(k) - v(k+1)$ for k in $\mathbb{N}$. This classical homeomorphism maps the subspace of probability measures with finite first moment onto $\mathcal{V} \cap \ell_1$, corresponding to a finite number of customers in the network. The symmetry structure implies that these processes are Markov.

1.2 Laws of large numbers

Let c_0^0 and ℓ_p^0 for $p \geq 1$ denote the subspaces of sequences vanishing at 0 of the classical sequence spaces c_0 (with limit 0) and ℓ_p (with summable absolute p-th power). We define mappings with values in c_0^0 given for v in c_0 by

$$F_+(v)(k) = \alpha\big(v(k-1)^L - v(k)^L\big)\,, \quad F_-(v)(k) = \beta(v(k) - v(k+1))\,, \qquad k \geq 1\,, \tag{1.1}$$

and $F = F_+ - F_-$, and the nonlinear differential equation $\dot{u} = F(u)$ on $\mathcal{V}$, explicitly for $t \geq 0$

$$\dot{u}_t(k) = F(u_t)(k) = \alpha\big(u_t(k-1)^L - u_t(k)^L\big) - \beta(u_t(k) - u_t(k+1))\,, \qquad k \geq 1\,. \tag{1.2}$$

This corresponds to (1.6) in Vvedenskaya et al. [9] (with arrival rate λ and service rate 1) and (3.9) in Graham [2] (with arrival rate ν and service rate λ). Theorem 1 (a) in [9] and Theorem 3.3 in [2] yield that there exists a unique solution $u = (u_t)_{t\geq 0}$ taking values in $\mathcal{V}$ for (1.2), which is continuous, and if u_0 is in $\mathcal{V} \cap \ell_1$ then u takes values in $\mathcal{V} \cap \ell_1$.

A functional law of large numbers (LLN) for converging initial data follows from Theorem 2 in [9]. We give below a result contained in Theorem 3.4 in [2].

Theorem 1.1 *Let $(R_0^N)_{N\geq L}$ converge in probability to u_0 in $\mathcal{V}$. Then $(R^N)_{N\geq L}$ converges in probability in $\mathbb{D}(\mathbb{R}_+, \mathcal{V})$ to the unique solution $u = (u_t)_{t\geq 0}$ starting at u_0 for (1.2).*

The networks are stable for $\rho = \alpha/\beta < 1$. Then Theorem 1 (b) in Vvedenskaya et al. [9] yields that (1.2) has a globally stable point $\tilde{u}$ in $\mathcal{V} \cap \ell_1$ given by $\tilde{u}(k) = \rho^{(L^k-1)/(L-1)}$. A functional LLN in equilibrium for $(R^N)_{N\geq L}$ with limit $\tilde{u}$ follows by a compactness-uniqueness method validating the inversion of limits for large sizes and large times, see Theorem 5 in [9] and Theorem 4.4 in [2].

The results of [9] are extended in Graham [2], in particular to LLNs and propagation of chaos results on path space. Theorem 3.5 in [2] gives convergence bounds in variation norm for the chaoticity result on $[0,T]$ for $(X_i^N)_{1\leq i\leq N}$ for $(X_i^N(0))_{1\leq i\leq N}$ i.i.d. of law q, using results in Graham and

Méléard [4]. These bounds can be somewhat extended for initial data satisfying appropriate a priori controls, but behave exponentially badly for large T.

1.3 Central limit theorems

Graham [3] and the present paper seek asymptotically tight rates of convergence and confidence intervals, and study the fluctuations around the LLN limits. For R_0^N in $\mathcal{V}^N$ and u_0 in $\mathcal{V}$ we consider the process R^N, the solution u for (1.2), and the process $Z^N = (Z_t^N)_{t\geq 0}$ with values in c_0^0 given by

$$Z^N = \sqrt{N}(R^N - u)\,. \tag{1.3}$$

Graham [3] focuses on the *stationary* regime for $\alpha < \beta$ defining the initial data *implicitly*: the law of R_0^N is the invariant law for R^N and $u_0 = \tilde{u}$. The main result in [3] is Theorem 2.12, a functional central limit theorem (CLT) in equilibrium for $(Z^N)_{N\geq L}$ with limit a stationary Ornstein-Uhlenbeck process. This *implies* a CLT under the invariant laws for $(Z_0^N)_{N\geq L}$ with limit the invariant law for this Gaussian process, an important result which seems very difficult to obtain directly. The proofs actually involve appropriate *transient* regimes, ergodicity, and fine studies of the long-time behaviors, in particular a global exponential stability result for the nonlinear dynamical system (1.2) using intricate comparisons with linear equations and their spectral theory.

We complete here the study in [3] and derive a functional CLT in relation to Theorem 1.1, for the Skorokhod topology on Hilbert spaces with the weak topology, for a wide class of R_0^N and u_0 under the *assumption* that $(Z_0^N)_{N\geq L}$ converges in law (for instance satisfies a CLT). This covers without constraints on α and β many transient regimes with *explicit* initial conditions, such as i.i.d. queues with common law appropriately converging as N grows. Section 2 introduces in turn the main notions and results, and Section 3 leads progressively to the proof of the functional CLT by a compactness-uniqueness method.

2 The functional central limit theorem

For a sequence $w = (w(k))_{k\geq 1}$ such that $w(k) > 0$ we define the Hilbert spaces

$$L_2(w) = \left\{ x \in \mathbb{R}^{\mathbb{N}} : x(0) = 0\,,\ \|x\|_{L_2(w)}^2 = \sum_{k\geq 1} \left(\frac{x(k)}{w(k)}\right)^2 w(k) = \sum_{k\geq 1} x(k)^2 w(k)^{-1} < \infty \right\}$$

of which the elements are considered as measures identified with their densities with respect to the reference measure w. Then $L_1(w) = \ell_1^0$ and if w is summable then $\|x\|_1 \leq \|w\|_1^{1/2}\|x\|_{L_2(w)}$ and $L_2(w) \subset \ell_1^0$. For bounded w we have the Gelfand triplet $L_2(w) \subset \ell_2^0 \subset L_2(w)^* = L_2(w^{-1})$.

Also, $L_2(w)$ is an ℓ_2 space with weights, and we consider the ℓ_1 space with same weights

$$\ell_1(w) = \left\{ x \in \mathbb{R}^{\mathbb{N}} : x(0) = 0\,,\ \|x\|_{\ell_1(w)} = \sum_{k\geq 1} |x(k)| w(k)^{-1} < \infty \right\}.$$

Clearly $x \in L_2(w) \Leftrightarrow x^2 \in \ell_1(w)$. The operator norm of the inclusion $\mathcal{V} \cap \ell_1(w) \hookrightarrow \mathcal{V} \cap L_2(w)$ is bounded by 1 since $\|x\|^2_{L_2(w)} = \|x^2\|_{\ell_1(w)} \leq \|x\|_{\ell_1(w)}$ for $\|x\|_\infty \leq 1$.

In the sequel we assume that $w = (w_k)_{k\geq 1}$ satisfies the condition that

$$\exists\, c, d > 0 \,:\ cw(k+1) \leq w(k) \leq dw(k+1) \text{ for } k \geq 1\,. \tag{2.1}$$

This holds for $\theta > 0$ for the geometric sequence $(\theta^k)_{k\geq 1}$, yielding quite strong norms for $\theta < 1$.

Theorem 2.1 *Let w satisfy (2.1). Then in $\mathcal{V}$ the mappings F, F_+ and F_- are Lipschitz for the $L_2(w)$ and the $\ell_1(w)$ norms. Existence and uniqueness holds for (1.2) in $\mathcal{V} \cap L_2(w)$ and in $\mathcal{V} \cap \ell_1(w)$.*

Proof. We give the proof for $\ell_1(w)$, the proof for $L_2(w)$ being similar (see Theorem 2.2 in Graham [3]). The identity $x^L - y^L = (x-y)(x^{L-1} + x^{L-2}y + \cdots + y^{L-1})$ yields

$$\left|u(k-1)^L - v(k-1)^L\right| w(k)^{-1} \leq |u(k-1) - v(k-1)|\, Ldw(k-1)^{-1}\,,$$

$$\left|u(k)^L - v(k)^L\right| w(k)^{-1} \leq |u(k) - v(k)|\, Lw(k)^{-1}\,,$$

$$|u(k+1) - v(k+1)|\, w(k)^{-1} \leq |u(k+1) - v(k+1)|\, c^{-1}w(k+1)^{-1}\,,$$

hence $\|F_+(u) - F_+(v)\|_{\ell_1(w)} \leq \alpha L(d+1)\|u-v\|_{\ell_1(w)}$ and $\|F_-(u) - F_-(v)\|_{\ell_1(w)} \leq \beta(c^{-1} + 1)\|u - v\|_{\ell_1(w)}$. Existence and uniqueness follows using a Cauchy-Lipschitz method. $\square$

The linearization of (1.2) around a particular solution u in $\mathcal{V}$ is the linearization of the equation satisfied by $z = g - u$ where g is a generic solution for (1.2) in $\mathcal{V}$, and is given for $t \geq 0$ by

$$\dot{z}_t = \mathbf{K}(u_t)z_t \tag{2.2}$$

where for v in $\mathcal{V}$ the linear operator $\mathbf{K}(v) : x \mapsto \mathbf{K}(v)x$ on c_0^0 is given by

$$\mathbf{K}(v)x(k) = \alpha Lv(k-1)^{L-1}x(k-1) - (\alpha Lv(k)^{L-1} + \beta)x(k) + \beta x(k+1)\,, \quad k \geq 1\,. \tag{2.3}$$

The infinite matrix in the canonical basis $(0,1,0,0\ldots)$, $(0,0,1,0\ldots)$, $\ldots$ is given by

$$\begin{pmatrix}
-\left(\alpha Lv(1)^{L-1} + \beta\right) & \beta & 0 & \cdots \\
\alpha Lv(1)^{L-1} & -\left(\alpha Lv(2)^{L-1} + \beta\right) & \beta & \cdots \\
0 & \alpha Lv(2)^{L-1} & -\left(\alpha Lv(3)^{L-1} + \beta\right) & \cdots \\
0 & 0 & \alpha Lv(3)^{L-1} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and $\mathbf{K}(v)$ is the adjoint of the the infinitesimal generator of a sub-Markovian birth and death process. The spectral representation of Karlin and McGregor [7] was a key tool in Graham [3], but here it varies in time and introduces no true simplification.

Let $(M(k))_{k\in\mathbb{N}}$ be independent real continuous centered Gaussian martingales, determined in law by their deterministic Doob-Meyer brackets given for $t \geq 0$ by

$$\langle M(k)\rangle_t = \int_0^t \left\{F_+(u_s)(k) + F_-(u_s)(k)\right\} ds\,. \tag{2.4}$$

The processes $M = (M(k))_{k\geq 0}$ and $\langle M\rangle = (\langle M(k)\rangle)_{k\in\mathbb{N}}$ have values in c_0^0.

Theorem 2.2 *Let w satisfy (2.1) and u_0 be in $\mathcal{V}\cap\ell_1(w)$. Then the Gaussian martingale M is square-integrable in $L_2(w)$.*

Proof. We have $\mathbf{E}\big(\|M_t\|^2_{L_2(w)}\big) = \mathbf{E}\big(\|\langle M\rangle_t\|_{\ell_1(w)}\big)$ and we conclude using (2.4), Theorem 2.1, and uniform bounds in $\ell_1(w)$ on $(u_s)_{0\leq s\leq t}$ in function of u_0 given by the Gronwall Lemma. $\square$

The limit Ornstein-Uhlenbeck equation for the fluctuations is the inhomogeneous affine SDE given for $t \geq 0$ by

$$Z_t = Z_0 + \int_0^t \mathbf{K}(u_s)Z_s\,ds + M_t \tag{2.5}$$

which is a perturbation of (2.2). A well-defined solution is called an Ornstein-Uhlenbeck process.

In equilibrium $u = \tilde{u}$ and setting $\mathcal{K} = \mathbf{K}(\tilde{u})$ and using (1.1) and $F_+(\tilde{u}) = F_-(\tilde{u})$ yields the simpler and more explicit formulation in Section 2.2 in Graham [3]. We recall that strong (or pathwise) uniqueness implies weak uniqueness, and that $\ell_1(w) \subset L_2(w)$.

Theorem 2.3 *Let the sequence w satisfy (2.1).*

(a) For v in $\mathcal{V}$, the operator $\mathbf{K}(v)$ is bounded in $L_2(w)$, and its operator norm is uniformly bounded.

(b) Let u_o be in $\mathcal{V}\cap L_2(w)$. Then in $L_2(w)$ there is a unique solution $z_t = \mathrm{e}^{\int_0^t \mathbf{K}(u_s)\,ds}z_0$ for (2.2) and strong uniqueness of solutions holds for (2.5).

(c) Let u_o be in $\mathcal{V}\cap\ell_1(w)$. Then in $L_2(w)$ there is a unique strong solution $Z_t = \mathrm{e}^{\int_0^t \mathbf{K}(u_s)\,ds}Z_0 + \int_0^t \mathrm{e}^{\int_s^t \mathbf{K}(u_r)\,dr}dM_s$ for (2.5) and if $\mathbf{E}\Big(\|Z_0\|^2_{L_2(w)}\Big) < \infty$ then $\mathbf{E}\Big(\sup_{t\leq T}\|Z_t\|^2_{L_2(w)}\Big) < \infty$.

Proof. Considering (2.3), $v \leq 1$, convexity bounds, and (2.1), we have

$$\begin{aligned}
\|\mathbf{K}(v)x\|^2_{L_2(w)} &\leq 2(\alpha L+\beta)\sum_{k\geq 1}\big(\alpha L x(k-1)^2 d w(k-1)^{-1} + (\alpha L+\beta)x(k)^2 w(k)^{-1} \\
&\qquad\qquad + \beta x(k+1)^2 c^{-1} w(k+1)^{-1}\big) \\
&\leq 2(\alpha L+\beta)(\alpha L(d+1) + \beta(c^{-1}+1))\|x\|^2_{L_2(w)}
\end{aligned}$$

and (a) and (b) follow, the Gronwall Lemma yielding uniqueness. Under the assumption on u_0 in (c) the martingale M is square-integrable in $L_2(w)$. If $\mathbf{E}\Big(\|Z_0\|^2_{L_2(w)}\Big) < \infty$ then the formula for Z is well-defined, solves the SDE, and the Gronwall Lemma yields $\mathbf{E}\Big(\sup_{t\le T}\|Z_t\|^2_{L_2(w)}\Big) < \infty$, else for any $\varepsilon > 0$ we can find $r_\varepsilon < \infty$ such that $\mathbf{P}\big(\|Z_0\|_{L_2(w)} \le r_\varepsilon\big) > 1-\varepsilon$, and a localization procedure using pathwise uniqueness yields existence. $\square$

Our main result is the following functional CLT. We refer to Jakubowski [5] for the Skorokhod topology for non-metrizable topologies. For the weak topology of a reflexive Banach space, the relatively compact sets are the bounded sets for the norm, see Rudin [8] Theorems 1.15 (b), 3.18, and 4.3. Hence, $B(r)$ denoting the closed ball centered at 0 of radius r, a set $\mathcal{T}$ of probability measures is tight if and only if for all $\varepsilon > 0$ there exists $r_\varepsilon < \infty$ such that $p(B(r_\varepsilon)) > 1-\varepsilon$ uniformly for p in $\mathcal{T}$, which is the case if $\mathcal{T}$ is finite.

Theorem 2.4 *Let w satisfy (2.1). Consider $L_2(w)$ with its weak topology and $\mathbb{D}(\mathbb{R}_+, L_2(w))$ with the corresponding Skorokhod topology. Let u_0 be in $\mathcal{V} \cap \ell_1(w)$ and R_0^N be in $\mathcal{V}^N$. Consider Z^N given by (1.3). If $(Z_0^N)_{N\ge L}$ converges in law to Z_0 in $L_2(w)$ and is tight, then $(Z^N)_{N\ge L}$ converges in law to the unique Ornstein-Uhlenbeck process solving (2.5) starting at Z_0 and is tight.*

3 The proof

Let $(x)_k = x(x-1)\cdots(x-k+1)$ for $x \in \mathbb{R}$ (the falling factorial of degree $k \in \mathbb{N}$). Let the mappings F_+^N and F^N and with values in c_0^0 be given for v in c_0 by

$$F_+^N(v)(k) = \alpha\,\frac{(Nv(k-1))_L - (Nv(k))_L}{(N)_L}\,, \quad k \ge 1\,, \qquad F^N(v) = F_+^N(v) - F_-(v)\,, \tag{3.1}$$

where F_- is given in (1.1). The process R^N is Markov on $\mathcal{V}^N$, and when in state r has jumps in its k-th coordinate, $k \ge 1$, of size $1/N$ at rate $NF_+^N(r)(k)$ and size $-1/N$ at rate $NF_-(r)(k)$.

Lemma 3.1 *Let R_0^N be in $\mathcal{V}^N$, u solve (1.2) starting at u_0 in $\mathcal{V}$, and Z^N be given by (1.3). Then*

$$Z_t^N = Z_0^N + \int_0^t \sqrt{N}\,\big(F^N(R_s^N) - F(u_s)\big)\,ds + M_t^N \tag{3.2}$$

defines an independent family of square-integrable martingales $M^N = (M^N(k))_{k\in\mathbb{N}}$ independent of Z_0^N with Doob-Meyer brackets given by

$$\big\langle M^N(k)\big\rangle_t = \int_0^t \big\{F_+^N(R_s^N)(k) + F_-(R_s^N)(k)\big\}\,ds\,. \tag{3.3}$$

Proof. This follows from a classical application of the Dynkin formula. $\square$

The first lemma below shows that it is indifferent to choose the L queues with or without replacement at this level of precision, the second one is a linearization formula.

Lemma 3.2 *For $N \geq L \geq 1$ and a in $\mathbb{R}$ we have*

$$A^N(a) := \frac{(Na)_L}{(N)_L} - a^L = \sum_{j=1}^{L-1} (a-1)^j a^{L-j} \sum_{1 \leq i_1 < \cdots < i_j \leq L-1} \frac{i_1 \cdots i_j}{(N-i_1)\cdots(N-i_j)}$$

and $A^N(a) = N^{-1}O(a)$ uniformly for a in $[0,1]$.

Proof. We develop $\frac{(Na)_L}{(N)_L} = \prod_{i=0}^{L-1} \frac{Na-i}{N-i} = \prod_{i=0}^{L-1} \left(a + (a-1)\frac{i}{N-i}\right)$ to obtain the identity for $A^N(a)$ and we deduce easily from it that it is $N^{-1}O(a)$ uniformly for a in $[0,1]$. $\square$

Lemma 3.3 *For $L \geq 1$ and a and h in $\mathbb{R}$ we have*

$$B(a,h) := (a+h)^L - a^L - La^{L-1}h = \sum_{i=2}^{L} \binom{L}{i} a^{L-i} h^i$$

with $B(a,h) = 0$ for $L = 1$ and $B(a,h) = h^2$ for $L = 2$. For $L \geq 2$ we have $0 \leq B(a,h) \leq h^L + \left(2^L - L - 2\right) ah^2$ for a and $a+h$ in $[0,1]$.

Proof. The identity is Newton's binomial formula. A convexity argument yields $B(a,h) \geq 0$. For a and $a+h$ in $[0,1]$ and $L \geq 2$, $B(a,h) \leq h^L + \sum_{i=2}^{L-1} \binom{L}{i} ah^2 = h^L + \left(2^L - L - 2\right) ah^2$. $\square$

For v in $\mathcal{V}$ and x in c_0^0, considering (1.1), (3.1) and Lemma 3.2 let $G^N : \mathcal{V} \to c_0^0$ be given by

$$G^N(v)(k) = \alpha A^N(v(k-1)) - \alpha A^N(v(k)) , \quad k \geq 1 , \tag{3.4}$$

and considering (1.1), (2.3) and Lemma 3.3 let $H : \mathcal{V} \times c_0^0 \to c_0^0$ be given by

$$H(v,x)(k) = \alpha B(v(k-1), x(k-1)) - \alpha B(v(k), x(k)) , \quad k \geq 1 \tag{3.5}$$

so that for $v + x$ in $\mathcal{V}$

$$F^N = F + G^N , \qquad F(v+x) - F(v) = \mathbf{K}(v)x + H(v,x) , \tag{3.6}$$

and we derive the limit equation (2.5) and (2.4) for the fluctuations from (3.2) and (3.3).

Lemma 3.4 *Let w satisfy (2.1). Let u_0 be in $\mathcal{V} \cap \ell_1(w)$ and R_0^N be in $\mathcal{V}^N$. For $T \geq 0$ we have*

$$\limsup_{N\to\infty} \mathbf{E}\left(\left\|Z_0^N\right\|_{L_2(w)}^2\right) < \infty \Rightarrow \limsup_{N\to\infty} \mathbf{E}\left(\sup_{0 \leq t \leq T} \left\|Z_t^N\right\|_{L_2(w)}^2\right) < \infty .$$

Proof. Using (3.2) and (3.6)

$$Z_t^N = Z_0^N + M_t^N + \sqrt{N} \int_0^t G^N(R_s^N)\, ds + \int_0^t \sqrt{N} \left(F(R_s^N) - F(u_s)\right) ds \tag{3.7}$$

where Lemma 3.2 and (2.1) yield that $G^N(R_s^N)(k) = N^{-1}O\left(R_s^N(k-1) + R_s^N(k)\right)$ and

$$\left\|G^N(R_s^N)\right\|_{L_2(w)} = N^{-1}O\left(\left\|R_s^N\right\|_{L_2(w)}\right) . \tag{3.8}$$

We have

$$\left\|R_s^N\right\|_{L_2(w)} \le \|u_s\|_{L_2(w)} + N^{-1/2}\left\|Z_s^N\right\|_{L_2(w)}, \tag{3.9}$$

Theorem 2.1 yields that F_+, F_- and F are Lipschitz, the Gronwall Lemma that for some $K_T < \infty$

$$\sup_{0\le t\le T}\left\|Z_t^N\right\|_{L_2(w)} \le K_T\left(\left\|Z_0^N\right\|_{L_2(w)} + N^{-1/2}K_T\|u_0\|_{L_2(w)} + \sup_{0\le t\le T}\left\|M_t^N\right\|_{L_2(w)}\right),$$

and we conclude using the Doob inequality, (3.3), (3.6),

$$\left\|F_+(R_s^N) + F_-(R_s^N)\right\|_{L_2(w)} \le K\left\|R_s^N\right\|_{L_2(w)}, \tag{3.10}$$

and the bounds (3.8) and (3.9). □

Lemma 3.5 *Let w satisfy (2.1), and consider $L_2(w)$ with its weak topology and $\mathbb{D}(\mathbb{R}_+, L_2(w))$ with the corresponding Skorokhod topology. Let u_0 be in $\mathcal{V}\cap\ell_1(w)$ and R_0^N be in $\mathcal{V}^N$. Consider Z^N given by (1.3). If $(Z_0^N)_{N\ge L}$ is tight then $(Z^N)_{N\ge L}$ is tight and its limit points are continuous.*

Proof. For $\varepsilon > 0$ let $r_\varepsilon < \infty$ be such that $\mathbf{P}(Z_0^N \in B(r_\varepsilon)) > 1-\varepsilon$ for $N \ge 1$ (see the discussion prior to Theorem 2.4). Let $R_0^{N,\varepsilon}$ be equal to R_0^N on $\{Z_0^N \in B(r_\varepsilon)\}$ and such that $Z_0^{N,\varepsilon}$ is uniformly bounded in $L_2(w)$ on $\{Z_0^N \notin B(r_\varepsilon)\}$ (for instance deterministically equal to some outcome of R_0^N on $\{Z_0^N \in B(r_\varepsilon)\}$). Then $Z_0^{N,\varepsilon}$ is uniformly bounded in $L_2(w)$ and we may use a coupling argument to construct $Z^{N,\varepsilon}$ and Z^N coinciding on $\{Z_0^N \in B(r_\varepsilon)\}$.

Hence to prove tightness of $(Z^N)_{N\ge L}$ we may restrict our attention to $(Z_0^N)_{N\ge L}$ uniformly bounded in $L_2(w)$, for which we may use Lemma 3.4.

The compact subsets of $L_2(w)$ are Polish, a fact yielding tightness criteria. We deduce from Theorems 4.6 and 3.1 in Jakubowski [5], which considers completely regular Hausdorff spaces (Tychonoff spaces) of which $L_2(w)$ with its weak topology is an example, that $(Z^N)_{N\ge L}$ is tight if

1. For each $T \ge 0$ and $\varepsilon > 0$ there is a bounded subset $K_{T,\varepsilon}$ of $L_2(w)$ such that for $N \ge L$ we have $\mathbf{P}\big(Z^N \in \mathbb{D}([0,T], K_{T,\varepsilon})\big) > 1-\varepsilon$.

2. For each $d \ge 1$, the d-dimensional processes $(Z^N(1), \dots, Z^N(d))_{N\ge L}$ are tight.

Lemma 3.4 and the Markov inequality yield condition 1. We use (3.7) (derived from (3.2)) and (3.3) and (3.6), and the bounds (3.8), (3.9) and (3.10). The uniform bounds in Lemma 3.4 and the fact that $Z^N(k)$ has jumps of size $N^{-1/2}$ classically imply that the above finite-dimensional processes are tight and have continuous limit points, see for instance Ethier-Kurtz [1] Theorem 4.1 p. 354 or Joffe-Métivier [6] Proposition 3.2.3 and their proofs. □

End of the proof of Theorem 2.4. Lemma 3.5 implies that from any subsequence of Z^N we may extract a further subsequence which converges to some Z^∞ with continuous sample paths. Necessarily Z_0^∞ has same law as Z_0. In (3.7) we have considering (3.6)

$$\sqrt{N}\big(F(R_s^N)(k) - F(u_s)(k)\big) = \mathbf{K}(u_s)Z_s^N + \sqrt{N}H\big(u_s, N^{-1/2}Z_s^N\big)\,. \tag{3.11}$$

We use the bounds (3.8), (3.9) and (3.10), the uniform bounds in Lemma 3.4, and additionally (3.5) and Lemma 3.3. We deduce by a martingale characterization that Z^∞ has the law of the Ornstein-Uhlenbeck process unique solution for (2.5) in $L_2(w)$ starting at Z_0^∞, see Theorem 2.3; the drift vector is given by the limit for (3.2) and (3.7) considering (3.11), and the martingale bracket by the limit for (3.3). See for instance Ethier-Kurtz [1] Theorem 4.1 p. 354 or Joffe-Métivier [6] Theorem 3.3.1 and their proofs for details. Thus, this law is the unique accumulation point for the relatively compact sequence of laws of $(Z^N)_{N\geq 1}$, which must then converge to it, proving Theorem 2.4.

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